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Classical statistical distributions can violate Bell-type inequalities

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Abstract
We investigate two-particle phase-space distributions in classical mechanics constructed to be the analogs of quantum-mechanical angular-momentum eigenstates. We obtain the phase-space averages of specific observables related to the projection of the particles' angular momentum along axes with different orientations, and show that the ensuing correlation function violates Bell's inequality. The key to the violation resides in choosing observables impeding the realization of the joint measurements whose existence is required in the derivation of the inequalities. This situation can have statistical (detection related) or dynamical (interaction related) underpinnings, but non-locality does not play any role.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Bell's theorem was originally introduced [1] to examine quantitatively the consequences of postulating hidden variable distributions on the incompleteness of quantum mechanics put forward by Einstein, Podolsky and Rosen [2] (EPR). The core of the theorem takes the form of inequalities involving average values of two observables each related to one of the two particles. Bell showed that these inequalities must be satisfied by any theory containing local variables aiming to complete quantum mechanics in the EPR sense. The assumptions leading to Bell's theorem imply the existence of a joint probability distribution accounting for the simultaneous existence of incompatible quantum observables [3, 4]. Local models forbidding the existence of these joint distributions are therefore not bound by Bell's theorem. Indeed local models violating Bell-type inequalities have already been proposed [5–7], but these models are mathematical and abstract. In this work, we show that the familiar statistical distributions of classical mechanics may lead to a violation of the relevant Bell-type inequalities. Our main ingredients will consist first in choosing specific classical phase-space ensembles for two particles (distributions constructed to be the classical analogs of the quantum-mechanical angular momenta eigenstates), and then in choosing detectors impeding the existence of the joint probability distribution. We will consider two types of settings involving angular-momentum measurements, each setting being closely related to a well-known quantum-mechanical context. The first setting will consist of a classical version of the detection loop hole [8]: the relevant Bell inequalities will be seen to be violated when the sampling is done on a sub-ensemble, defined by the type of detected events, leading to averages computed on a partial region of phase space over which the joint probability distribution cannot be defined. Hence the violation has a statistical underpinning—we will show there is no violation if the averages are taken on the entire phase space. The second setting will reveal a genuine violation of the Bell inequalities due to dynamical reasons: by including in the detection process a local probabilistic interaction between the measured particle and the detector inducing a random perturbation with a constraint that blurs the particles' individual phase-space positions, the derivation of Bell's theorem is effectively blocked, as only correlations between ensembles corresponding to a fixed setting of the detectors can be made. This example can be seen as a classical version of the quantum measurement of non-commuting observables.

2. Classical ensembles

We first introduce the classical analogs of the quantum-mechanical angular-momentum eigenstates to be employed below. The classical distributions of particles can be considered either in phase space or in configuration space; equivalently, one can also consider the distribution of the angular momenta on the angular-momentum sphere. Let us first take a single classical particle and assume that the modulus J of its angular momentum is fixed. The value of $\mathbf{J}$ then depends on the position of the particle in the phase space defined by $\Omega = \{\theta, \phi, p_\theta, p_\phi\}$, where θ and ϕ refer to the polar and azimuthal angles in spherical coordinates and p_θ and p_ϕ are the conjugate canonical momenta. Let $\rho_z(\Omega)$ be the distribution in phase space given by

$$\rho_{z_0}(\theta, \phi, p_\theta, p_\phi) = N\delta(J_z(\Omega) - J_{z0})\delta\big(J^2(\Omega) - J_0^2\big). \tag{1}$$

ρ_{z_0} defines a distribution in which every particle has an angular momentum with the same magnitude, namely J_0, and the same projection on the z axis J_{z_0}. Hence ρ_{z_0} can be considered as a classical analog of the quantum-mechanical density matrix $|jm\rangle\langle jm|$ since just like a quantum measurement of the magnitude j and of the z axis projection m of the angular momentum in such a state will invariably yield the eigenvalues of the operators $\hat{J}^2$ and $\hat{J}_z$, the classical measurement of these quantities when the phase-space distribution is known to be ρ_z will give J_0^2 and J_{z_0} (see Appendix A). Equation (1) can be integrated over the conjugate momenta to yield the *configuration space* distribution

$$\rho(\theta, \phi) = N\big[\sin(\theta)\sqrt{J_0^2 - J_{z_0}^2 / \sin^2(\theta)}\big]^{-1}, \tag{2}$$

where we have used the defining relations $J_z(\Omega) = p_\phi$ and $J^2(\Omega) = p_\theta^2 + p_\phi^2 / \sin^2\theta$. Further integrating over θ and ϕ and requiring the phase-space integration of ρ to be unity allows one to set the normalization constant $N = J_0/2\pi^2$.

There is of course nothing special about the z axis and we can define a distribution by fixing the projection J_a of the angular momentum on an arbitrary axis a to be constant (in this paper we will take all the axes to lie in the zy plane). Computing the distribution $\rho_{a_0} = \delta(J_a - J_{a_0})\delta\big(J - J_0^2\big)$ is tantamount to rotating the coordinates towards the a axis in equation (2). Figure 1 shows examples of configuration space particle distributions and gives for one plot the corresponding quantum-mechanical angular-momentum eigenstate

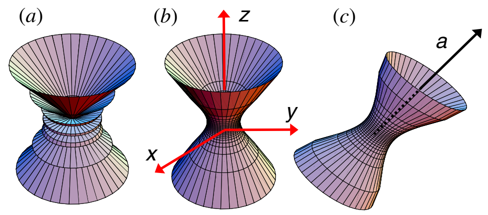

Figure 1. Normalized angular distribution for a single particle in configuration space. (a) *Quantum* distribution (spherical harmonic $|Y_{JM}(\theta,\phi)|^2$). (b) *Classical* distribution $\rho_{z_0}(\theta,\phi)$ of equation (2). (c) Classical distribution ρ_{a_0} corresponding to a fixed value of J_a (here $\theta_a = \pi/4$). The angular momentum and the projection on the z [(a)–(b)] or a [(c)] axis is the same for the three plots ($J/\eta = 40$, with $\eta = \hbar$, and $M/J = 5/8$).

(the similarity is not accidental, as equation (2) is essentially the amplitude of the spherical harmonic in the semiclassical regime, see appendix A). We can also determine the average projection J_a on the a axis for a distribution of the type (2) corresponding to a well-defined value of J_z:

$$\langle J_a \rangle_{J_{z_0}} = \int p_\phi \cos\theta_a \delta(J_z(\Omega) - J_{z_0}) \mathrm{d}\Omega = J_{z_0} \cos\theta_a, \tag{3}$$

where θ_a is the angle $(\widehat{z,a})$ and the projection of the component of J_a on the y axis vanishes given the axial symmetry of the distribution.

The original derivation of the inequalities by Bell [1] involved the measurement of the angular momentum of two spin-1/2 particles along different axes. Here we will consider the fragmentation of an initial particle with a total angular momentum $\mathbf{J}_T = 0$ into two particles carrying angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$ (we will assume to be dealing with orbital angular momenta). Conservation of the total angular momentum imposes $\mathbf{J}_1 = -\mathbf{J}_2$ and $J_1 = J_2 \equiv J$. Quantum mechanically, this situation would correspond to the system being in the singlet state arising from the composition of the angular momenta ($j_T = 0, m_1 = -m_2$). Classically the system is represented by the 2-particle phase-space distribution

$$\rho(\Omega_1, \Omega_2) = N\delta(\mathbf{J}_1 + \mathbf{J}_2), \tag{4}$$

where N is again a normalization constant. On the angular-momentum sphere the distribution (4) corresponds to $\mathbf{J}_1$ and $\mathbf{J}_2$ being uniformly distributed on the sphere but pointing in opposite directions. This distribution will now be employed for determining averages of observables related to the angular momenta of the two particles.

3. Statistical violation of the Bell inequalities

Let us assume two types of detectors yielding outcomes related to the angular momenta of the particles. The first type gives a 'sharp' (S) measurement of J_{1a} only if J_{1a} is an integer multiple of some elementary gauge η, and gives 0 elsewhere. This detection can be represented by the phase-space quantity

$$S_a(\Omega_1) = J_{1a}(\Omega_1) \quad \text{if} \quad \Omega_1 \in \Omega_{1k}, \qquad S_a(\Omega_1) = 0 \text{ elsewhere}, \tag{5}$$

where Ω_{1k} are the parts of phase space where $J_{1a} = k\eta$ compatible with a detection (see figure 2(a)). The second detector gives a 'direct' (D) measurement of J_{2b} (the projection of $\mathbf{J}_2$ on an axis b). The corresponding phase-space function is

$$D_b(\Omega_2) = \mathbf{J}_2 \cdot \mathbf{b}. \tag{6}$$

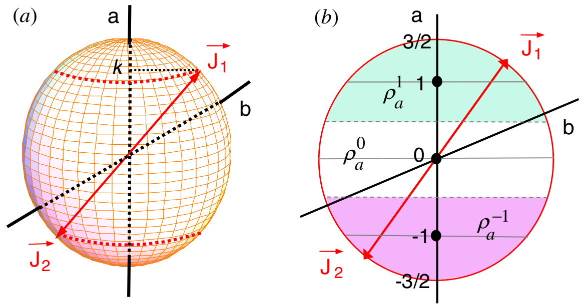

Figure 2. Setups for the first (*a*) and second (*b*) examples investigated in this work. In (*a*) an S detector is placed along the a axis and a D detector along b. The angular momenta, originally distributed on the sphere, are constrained to move on the rings (red dotted) corresponding to fixed values of the projection on a. (*b*) shows the $(\widehat{z,y})$ plane of the angular-momentum sphere for the $L = 1$ case (hence $J = 3/2$); the three zones correspond to the projections of the spherical zones ρ_a^{-1}, ρ_a^0 and ρ_a^{+1}. If the distribution corresponds to one of these ensembles, measuring J_a will yield respectively the outcomes $J_a = -1, 0, 1$ with unit probability. If J_b is measured and $\mathbf{J} \in \rho_a^k$ *any* of the outcomes $J_b = -1, 0, 1$ can be obtained with probabilities depending on the distribution ρ_a^k.

In classical mechanics there is no natural unit for quantities having the dimension of an action, so J and η can be expressed in terms of arbitrary units, and any physical result will depend only on the ratio J/η. We will assume for definiteness that η is chosen so that the extremal values $\pm J$ can be reached. J/η must hence be either an integer or a half-integer, the extremal values in dimensionless units being given by $\pm L \equiv \pm J/\eta$. For example if $\eta = 2J$, the measurement can only yield the extremal values $L = \pm 1/2$ ($\eta = J$ allows one to measure $\pm L = \pm 1$ and 0, $\eta = 2J/3$ allows $\pm L = \pm 3/2$ and $\pm(L-1) = \pm 1/2$ etc). Note that the particle label 1 or 2 can be attached to the detectors: indeed, we will call '1' the particle detected by S and '2' the particle detected by D.

The classical average $E(a, b) = \langle S_a D_b \rangle$ for joint measurements over the statistical distribution ρ can be computed from

$$E(a, b) = \int S_a(\Omega_1) D_b(\Omega_2) \rho(\Omega_1, \Omega_2)\, \mathrm{d}\Omega_1\, \mathrm{d}\Omega_2 \tag{7}$$

with equations (4), (5) and (6). Given the characteristics (5) of the S detection, equation (7) is actually a discrete sum over the parts of phase-space Ω_{1k} leading to the detection of $k\eta$; this can be written by including a delta function under the integral. Equation (4) imposes $\theta_2 = \pi - \theta_1$ and $\phi_2 = \pi + \phi_1$, and equation (7) becomes

$$E(a, b) = \frac{1}{2} \sum_{k=-L}^{k=L} \int [L \cos\theta_1] \delta(L\cos\theta_1 - k)[-L\cos\theta_1 \cos(\theta_b - \theta_a)] \sin\theta_1\, \mathrm{d}\theta_1, \tag{8}$$

where we have chosen the z axis to coincide with a to take advantage of the axial symmetry imposed by S_a (here the limiting procedure in the delta function is understated). The $\frac{1}{2}$ prefactor is the only nontrivial normalization factor (coming from the integration over θ_1). We obtain the average as

$$E(a, b) = -\tfrac{1}{6}(L+1)(2L+1)\cos(\theta_b - \theta_a), \tag{9}$$

which as expected depends solely on the ratio $J/\eta \equiv L$.

The correlation function employed in Bell's inequality can be obtained in the standard (or CHSH) form [9, 10]. We choose four axes a, b,a', b' (we can assume that an S detector is placed along a and a', and a D detector along b and b') and determine the average values for each of the four possible combinations involving an S and a D detector. The correlation function C relating the average values obtained for different orientation of the detectors' axes is

$$C(a,b,a',b') = (|E(a,b) - E(a,b')| + |E(a',b) + E(a',b')|)(L)^{-2}, \tag{10}$$

where we have divided by L^2 to obtain the CHSH function in the standard form characterized by values bounded by ± 1. Here the detected values obey the conditions $|S/L| \leqslant 1$ and $|D/L| \leqslant 1$, so that the usual derivation of the Bell inequalities would lead to

$$C(a,b,a',b') \leqslant 2. \tag{11}$$

By replacing equation (9) in equation (10), it can be seen that for $L = \frac{1}{2}$, 1 and $\frac{3}{2}$, there are several choices of the axes that lead to $C(a,b,a',b') > 2$. The maximal value of the correlation function corresponds to $C\left(0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right) = 4\sqrt{2}$ and $2\sqrt{2}$ for $L = \frac{1}{2}$ and 1 respectively[1].

The violation of the Bell inequality is due to the fact that we are only including in the statistics the measurements for which *both* the S and the D detectors click. But when an S-measurement is made along the two different orientations a and a' that enter the correlation function, different and mutually exclusive parts of phase space are selected, so that the different events

$$\{S_{1a}, D_{2b}\}, \{S_{1a'}, D_{2b}\}, \{S_{1a}, D_{2b'}\}, \quad \text{and} \quad \{S_{1a'}, D_{2b'}\} \tag{12}$$

are not supported by a common phase-space distribution. As a consequence the quantity

$$\int S_a(\Omega_1) D_b(\Omega_2) S_{a'}(\Omega_1) D_{b'}(\Omega_2) \rho(\Omega_1, \Omega_2)\, \mathrm{d}\Omega_1\, \mathrm{d}\Omega_2 \tag{13}$$

describing the average of simultaneous measurements along the four axes becomes undefined. However, as we mentioned above, the existence of the joint probability distribution in the integrand of equation (13), or equivalently [11], of a common distribution for the events (12) is a necessary ingredient in the derivation of Bell's theorem, thereby explaining the violation of the inequalities. It is noteworthy that if one includes the *entire* phase space in the average (7) instead of the parts of phase space corresponding to the double-click events, then equation (13) becomes well defined. It can then be shown that $E(a,b)$ and $C(a,b,a',b')$ should be multiplied by the fraction of phase-space yielding the double click measurements[2]: as a result Bell's inequality *would not be violated*. From the standpoint of classical mechanics, the objection regarding the necessity of including the entire phase space makes sense, since one can envisage in principle a particle analyzer able to detect the particles that have not been included in the double-click statistics. The quantum analog of this problem is the well-known detection loophole, pending on the experimental tests of Bell's inequalities [8, 13].

4. Dynamically induced violation of the Bell inequalities

Our second setting goes further into the violation of Bell's inequalities by postulating a model involving a local probabilistic interaction during the measurement between the detector and

[1] The reader familiar with the Bell inequalities for the quantum measurement of J_{1a} and J_{2b} will recognize the similarity of equation (9) with the quantum expectation value; the only difference is that the quantum expectation value is normalized respective to the number of possible outcomes $(2L+1)$ whereas here the normalization is relative to classical phase space (namely the length $2L$ of the measurement axis).

[2] Here this part of phase space is infinitesimal, since for the sake of mathematical simplicity we have modeled the S detection by a delta function. If we replace the delta functions on the angular-momentum sphere by narrow rings and spherical caps having a finite surface, the fraction of phase-space leading to double-click events becomes finite, and the reasoning as well as the conclusions reached with the delta function modeling hold (although the computations need to be made numerically) [12].

the particle being measured obeying a specific constraint: we then obtain a violation of the inequality for the entire ensemble of particles. Let us take two identical detectors T_1 and T_2 that give as only output the integer or half-integer values $k = L, L-1, \ldots - L$ of the projection J_{1a} and J_{2b} of the angular momenta of the particles. We choose here $L = J/\eta - 1/2$, from which it follows that the maximal readout L is smaller than J; for notational simplicity we put $\eta = 1$ (so J, rather than J/η takes integer or half integer values). We further assume that there is an interaction between T_1 and particle 1 (and between T_2 and particle 2) affecting the angular momentum of the particle in a specific way.

We impose the following constraints on this process (which only involves a *single* particle and its measuring apparatus, hence we drop the indices labeling the particles).

(i) There are distributions ρ_a^k such that if $\mathbf{J} \in \rho_a^k$

$$P_{k'}^{T_a}\left(\mathbf{J} \in \rho_a^k\right) = \delta_{kk'}. \tag{14}$$

This means that if T_a is measured and we obtain k then we know that previous to the measurement $\mathbf{J} \in \rho_a^k$ with unit probability.

(ii) Let $\langle J_b \rangle_{\rho_a^k}$ be the phase-space average of J_b over the distribution ρ_a^k, where the directions b and a are assumed to be different. If T_b is measured and $\mathbf{J} \in \rho_a^k$, *any* outcome k' can be obtained with a *non-vanishing* probability $P_{k'}^{T_b}\left(\mathbf{J} \in \rho_a^k\right)$. Our main assumption is that averaging over T_b gives the phase-space average of J_b, i.e. the interaction vanishes on average. This constraint takes the form

$$\langle T_b \rangle_{\rho_a^k} = \sum_{k'=-L}^{L} k' P_{k'}^{T_b}\left(\mathbf{J} \in \rho_a^k\right) = \langle J_b \rangle_{\rho_a^k}. \tag{15}$$

Equation (15) also holds if $b = a$ but then only $P_k^{T_a} = 1$ is non-vanishing hence

$$\langle T_a \rangle_{\rho_a^k} = \langle J_a \rangle_{\rho_a^k} = k. \tag{16}$$

We will not be interested here in putting forward specific models of the interaction yielding such probabilities; it will suffice for our purpose that a set of numbers P_k verifying equation (15) and obeying $\sum_k P_k = 1$ can be obtained. We need to specify however the distributions obeying equation (14). It is convenient to specify ρ_a^k in terms of the distribution of $\mathbf{J}$ on the angular-momentum sphere: it can then easily be seen that equation (16) is realized if ρ_a^k is taken to be the ring centered on the a axis and bounded by $k - 1/2 < J_a < k + 1/2$ (see figure 2(*b*)). Then a measurement of T_a will yield the outcome k with unit probability:

$$T_a = k \qquad \text{if} \quad k - 1/2 < J_a < k + 1/2. \tag{17}$$

One can of course envisage a distribution ρ obtained by combining the elementary ensembles ρ_a^k. In particular the uniform distribution on the sphere ρ_Σ is the sum of the $2L + 1$ spherical rings ρ_a^k,

$$\rho_\Sigma = \sum_k \frac{\rho_a^k}{2L + 1} \tag{18}$$

and therefore if T_a is measured the probability of finding a given value k is $P = 1/(2L + 1)$. Inversely the obtention of the given outcome k is correlated with $\mathbf{J} \in \rho_a^k$ previous to the measurement. With ρ_a^k defined in this way (equation (17)), $\langle J_b \rangle_{\rho_a^k}$ is computed straightforwardly and equation (15) becomes

$$\langle T_b \rangle_{\rho_a^k} = k \cos(\theta_b - \theta_a), \tag{19}$$

we see again that for correlations involving averages, the knowledge of the individual probabilities $P_k^{T_b}$ is not necessary. Note however that for the particular case $J = 1$ (i.e., $L = 1/2$) the constraints (15)–(17) as well as the normalization of the probabilities impose the values of the $P_k^{T_b}$ irrespective of any precise physical process: indeed k can only take the values $\pm - 1/2$ from which it follows that

$$P_{\pm}^{T_b} = \tfrac{1}{2} \pm \langle J_a \rangle_{\rho_a^k} = \tfrac{1}{2} \pm k \cos(\theta_b - \theta_a). \tag{20}$$

Let us now go back to the 2-particle problem, assuming the initial phase-space density ρ given by equation (4). The expectation value $E(a,b) = \langle T_{1a} T_{2b} \rangle$ is computed from the general formula

$$E(a,b) = \sum_{k,k'=-L}^{L} kk' P(T_{2b} = k' \cap T_{1a} = k), \tag{21}$$

where k and k' run on the possible outcomes. The probabilities of obtaining $T_{1a} = k$ and $T_{2b} = k'$ are obtained in the following way. Using

$$P(T_{2b} = k' \cap T_{1a} = k) = P(T_{1a} = k) P(T_{2b} = k' | T_{1a} = k) \tag{22}$$

we first determine $P(T_{1a} = k)$ by remarking that the initial distribution ρ corresponds to $\mathbf{J}_1$ being uniformly distributed on the sphere. According to the results of the preceding paragraph, with the sphere being cut into $2L + 1$ equiprobable zones ρ_a^k (see equation (18)), we have $P(T_{1a} = k) = 1/(2L + 1)$. We also know that an outcome $T_{1a} = k$ corresponds to $\mathbf{J}_1 \in \rho_a^k$ (equation (14)). From the conservation of the total angular momentum, we infer that particle 2 must lie in the zone ρ_a^{-k} defined by $k - 1/2 < -J_{2a} < k + 1/2$ (equation (17)); indeed if T_{2a} were to be measured we would be assured of finding $T_{2a} = -T_{1a} = -k$. Hence the conditional probability appearing in equation (22) is given by

$$P(T_{2b} = k' | T_{1a} = k) = P_{k'}^{T_b} \big(\mathbf{J}_2 \in \rho_a^{-k} \big), \tag{23}$$

where $P_{k'}^{T_b}$ was defined in equation (15). The sum over k' in equation (21) thus verifies equation (15) and having in mind equation (19), the expectation value becomes

$$E(a,b) = \sum_{k=-L}^{L} \frac{-k^2}{2L+1} \cos(\theta_b - \theta_a) = \frac{-L(L+1)}{3} \cos(\theta_b - \theta_a). \tag{24}$$

The correlation function is again given by equation (10), since the maximum value detected by a T measurement is L, not J. The result given by equation (24) is familiar from quantum mechanics—it violates Bell's inequality for $L = 1/2$ with a maximal violation for $C\big(0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\big) = 2\sqrt{2}$. As noted for the single-particle case, the derivation of $E(a,b)$ does not depend in any way on the individual values of the probabilities $P_{k'}^{T_b}$ but only on the condition (15) regarding the particle-measurement interaction. Note that by Bayes' theorem, it is of course equivalent to computing $P(T_{2b} = k' \cap T_{1a} = k)$ from $P(T_{2b} = k') P(T_{1a} = k | T_{2b} = k')$, i.e. by assuming that $T_{2b} = k'$ is known first.

The violation of the Bell inequalities is due to the conjunction of two ingredients. The first, represented by the constraints (14)–(16), is relative to a *single* particle and its interaction with the measurement apparatus. The second is the conservation of the angular momentum *on average*. Interestingly the first ingredient is the one that contradicts the assumptions made in the derivation of Bell's theorem. The reason is that equations (14)–(16) are incompatible with the introduction of elementary probability functions $p_k^{T_b}(\Omega)$ such that

$$P_{k'}^{T_b} \big(\mathbf{J} \in \rho_a^k \big) = \int p_{k'}^{T_b}(\Omega) \rho_a^k(\Omega) \, \mathrm{d}\Omega, \tag{25}$$

indeed, such probability functions would need to depend on the ensemble, giving rise to functions of the type $p_k^{T_b}(\Omega;\rho_a^k)$. This is shown for the case $L=1/2$ in appendix B. With this point in mind, one can expand equation (21) (with equations (22), (18) and (23)) as

$$E(a,b)=\int\sum_k kp_k^{T_a}(\Omega_1;\rho_\Sigma)\rho_\Sigma(\Omega_1)\,\mathrm{d}\Omega_1\int B(\Omega_2,k)\rho_a^{-k}(\Omega_2)\,\mathrm{d}\Omega_2 \qquad (26)$$

with

$$B(\Omega_2,k)\equiv\sum_{k'}k'p_{k'}^{T_b}\left(\Omega_2;\rho_a^{-k}\right). \qquad (27)$$

The dependence of B on k is the crucial property allowing to violate Bell's inequality (whereas the dependence of $\rho(\Omega_2)$ on k in equation (26) by itself can be absorbed in the initial correlation $\delta(\mathbf{J}_1+\mathbf{J}_2)$ provided $k=k(\Omega_1)$). The dependence of B on k has nothing to do with non-locality or action at a distance. It is a simple consequence of the logical inference characterizing the conditional probability (22) given the characteristics of the single particle interaction with the measuring apparatus, namely the fact that the model allows only specific types of correlations: in the *single* particle problem one can only correlate a given outcome with a specific distribution—this happens when the distribution is symmetric relative to the detector's axis (equation (14)); in the *two* particle problem the single particle property just mentioned makes only possible the correlation of $\mathbf{J}_2$ as a function of $\mathbf{J}_1$ in terms of the *ensembles* to which they belong, not in terms of their *individual* positions. This is consistent with the fact that the knowledge of the individual position of $\mathbf{J}$ is meaningless to compute the observed probabilities, as even the elementary probabilities must depend on the ensemble to which the angular momentum belongs[3].

Note finally that B would not depend on k (and the elementary probabilities on the ensembles), equation (26) would turn into

$$E^{\mathrm{BT}}(a,b)=\int A(\Omega_1)B(\Omega_2)\rho(\Omega_1,\Omega_2)\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2, \qquad (28)$$

where

$$A(\Omega_1)\equiv\sum_k kp_k^{T_a}(\Omega_1)\qquad B(\Omega_2)\equiv\sum_{k'}k'p_{k'}^{T_b}(\Omega_2), \qquad (29)$$

thereby yielding the familiar form taken by the expectation value in the derivation of Bell's theorem. In the *deterministic* case considered by Bell [10] the functions $p_k^{T_a}$ and $p_{k'}^{T_b}$ are either 0 or 1 depending on the individual position of $\mathbf{J}_1$ (respectively $\mathbf{J}_2$). This implies that $k=k(\Omega_1)$, i.e. a given outcome depends on the position of $\mathbf{J}_1$ on the angular-momentum sphere, and $\rho_a^{-k}(\Omega_2)=\rho(\Omega_2|\mathbf{J}_1)$ does not depend on k or a but on $\mathbf{J}_1=-\mathbf{J}_2$ (hence the inclusion of the term $\delta(\mathbf{J}_1+\mathbf{J}_2)$ in the definition of $\rho(\Omega_1,\Omega_2)$). Conversely one may *assume* $\rho(\Omega_2|k)=\rho(\Omega_2|\Omega_1)$ in equation (26) with $p_a^k(\Omega_1)$ and $p_b^{k'}(\Omega_2)$ being probability functions different from 0 or 1; then A and B defined in equation (29) are not the observed outcomes but their averages, and $E^{\mathrm{BT}}(a,b)$ is the expectation corresponding to the *stochastic* case considered by Bell. Bell's stochastic case correlates the individual positions of $\mathbf{J}_1$ and $\mathbf{J}_2$ to possible outcomes with definite probabilities. In the present model the random interaction forbids to make the correspondence between a given position of the angular momenta and a definite outcome; instead the correspondence is between a definite outcome and a given ensemble describing the positions of the angular momenta compatible with the outcome (of

[3] It would be of course extremely valuable to understand what kind of physical processes are compatible with this type of behavior (for example, the value of the angular momentum in this case could represent some time average of an underlying stochastic process, or a space average of a field-like quantity distributed all over the ensemble).

course if the former correspondence is satisfied, so is the latter, but the converse is not true). In the latter case, the structure of the expectation value (26) does not allow to define a term of the type given by equation (13) whereby a single distribution can account for several simultaneous joint measurements. It appears indeed that the ensemble dependence exhibited by the present model is a necessary feature in order to produce non-commuting measurements [12]. In this sense the present model can be seen as a classical analog of the quantum measurement of two non-commuting observables (such as J_{1a} and $J_{1a'}$) applied to correlations between two particles as originally considered by EPR [2].

5. Conclusion

The present results show that averages obtained with 2-particle classical distributions constructed to be the analogs of quantum-mechanical eigenstates can violate Bell's inequalities. The violation does not involve non-locality but statistical or dynamical processes that impede the existence of joint probability distributions or the correlation between individual values of the variables as required by Bell's theorem. Possible implications on the role of the Bell-CHSH argument as a marker of quantum nonlocality, which has recently been criticized [14], will be examined elsewhere [12].

Appendix A

The scheme we are employing to construct the classical distributions rests on the well-known analogy between the classical Poisson brackets and the quantum commutation relations in the density matrix formalism. Let $\hat{G}$ be an operator and $|\psi_g\rangle$ an eigenstate with eigenvalue g. Then the pure-state density matrix $\hat{\rho}_g \equiv |\psi_g\rangle\langle\psi_g|$ verifies $[\hat{\rho}_g, \hat{G}] = 0$ and $\hat{G}\hat{\rho}_g = g\hat{\rho}_g$. In classical mechanics the Poisson bracket of two phase-space quantities $u(q, p)$ and $v(q, p)$ is a canonical invariant defined by [15]

$$\{u, v\} = \frac{\partial u}{\partial q}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial q}. \tag{A.1}$$

Let $\rho(q, p)$ be the phase-space distribution and $G(q, p)$ be a function such that $\{\rho, G\} = 0$. This means that ρ is invariant relative to the canonical tranformation generated by G, i.e.

$$\{\rho, G\}\delta Q_G = \delta\rho = 0, \tag{A.2}$$

where Q_G is canonically conjugate to G, which is a constant of the motion. Then every point of the distribution ρ will be characterized by the constant value taken by G, denoted g. If this is the only constraint imposed on the distribution, $\rho(q, p)$ will take the form (up to a normalization constant)

$$\rho(q, p) = \delta(G(q, p) - g). \tag{A.3}$$

In configuration space, the distribution $\rho(q)$ is obtained by integrating over the values of the momentum compatible with a given q,

$$\rho(q) = \int \rho(q, p)\,\mathrm{d}p = \int \frac{\delta(p - p_i)}{\left.\frac{\partial G}{\partial p}\right|_{p_i}}\,\mathrm{d}p, \tag{A.4}$$

where p_i is the root (assumed to be unique, else a sum is in order) of the argument of the delta function. Integrating yields

$$\rho(q) = \left.\frac{\partial p}{\partial G}\right|_{p_i} = \left.\frac{\partial^2 S}{\partial q \partial G}\right|_{p_i}, \tag{A.5}$$

where $S(q, G)$ is the classical action. The configuration space density is therefore the amplitude of the quantum density matrix element $\langle q|\,\hat{\rho}_g|q\rangle$ in the semiclassical approximation.

Appendix B

We show that the detection model for a single particle given in section 4 is inconsistent with probability functions defined by equation (25) in the $L = 1/2$ case (the one violating the Bell inequalities). Take equation (25) with $a = b$ and $k, k' = +1/2$,

$$P_+^{T_b}(\mathbf{J} \in \rho_b^+) = \int p_+^{T_b}(\Omega)\rho_b^+(\Omega)\,\mathrm{d}\Omega = 1. \tag{B.1}$$

Particularizing the general formula (17) to the case $L = 1/2$, ρ_b^+ is the positive hemisphere of the unit sphere (since $J = 1$) centered on the b axis. The result on the right-hand side follows from equation (20). Equation (B.1) implies that $p_+^{T_b}(\Omega) = 1$ for $\mathbf{J} \in \rho_b^+$ and consequently $p_-^{T_b}(\Omega) = 0$. Conversely since $P_+^{T_b}(\mathbf{J} \in \rho_b^-) = 0$, we must have $p_+^{T_b}(\Omega) = 0$ and $p_-^{T_b}(\Omega) = 1$ when $\mathbf{J} \in \rho_b^-$. Now assume that the distribution is instead ρ_a^+ with a different from the b axis. Then according to our model (equation (20)) we should have

$$P_+^{T_b}(\mathbf{J} \in \rho_a^+) = \int p_+^{T_b}(\Omega)\rho_a^+(\Omega)\,\mathrm{d}\Omega = \cos^2\frac{\theta_b - \theta_a}{2}. \tag{B.2}$$

Noting that ρ_a^+, the positive hemisphere centered on a, is actually composed of two parts, $\rho_a^+ \cap \rho_b^+$ and $\rho_a^+ \cap \rho_b^-$ we can write

$$P_+^{T_b}(\mathbf{J} \in \rho_a^+) = \int_{\rho_a^+\cap\rho_b^+} p_+^{T_b}(\Omega)\rho_a^+(\Omega)\,\mathrm{d}\Omega + \int_{\rho_a^+\cap\rho_b^-} p_+^{T_b}(\Omega)\rho_a^+(\Omega)\,\mathrm{d}\Omega. \tag{B.3}$$

But we have seen that $p_+^{T_b} = 1$ for $\mathbf{J} \in \rho_b^+$ and $p_+^{T_b}(\Omega) = 0$ for $\mathbf{J} \in \rho_b^-$, hence

$$P_+^{T_b}(\mathbf{J} \in \rho_a^+) = \int_{\rho_a^+\cap\rho_b^+} \rho_a^+(\Omega)\,\mathrm{d}\Omega = 1 - \frac{\theta_b - \theta_a}{\pi}, \tag{B.4}$$

which contradicts equation (B.2). Hence probability functions obeying equation (25) do not exist, and equation (25) should be replaced by

$$P_{k'}^{T_b}\left(\mathbf{J} \in \rho_a^k\right) = \int p_{k'}^{T_b}\left(\Omega; \rho_a^k\right)\rho_a^k(\Omega)\,\mathrm{d}\Omega, \tag{B.5}$$

where the notation $p_{k'}^{T_b}(\Omega; \rho_a^k)$ denotes the dependence of the elementary probabilities on the distribution. Note also that equation (25) *does* hold if one drops the requirement that $p_{k'}^{T_b}(\Omega)$ should represent an elementary probability: for example the functions $p_+^{T_b}(\Omega) = J_b + 1/2$ or $p_+^{T_b}(\Omega) = 2J_b H(J_b)$ fulfil equation (B.2) without depending on the distribution, though none of these functions is contained in the interval [0, 1] and are thus not probability functions. We stress that these features, which put strong constraints on the type of admissible physical models that one could envisage, are relevant to a *single* particle and its interaction with the measurement apparatus.

References

[1] Bell J S 1964 *Physics* **1** 195
[2] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[3] Fine A 1982 *J. Math. Phys.* **23** 1306
[4] Cabello A and Garcia-Alcaine G 1997 *J. Phys. A: Math. Gen.* **30** 725
[5] Beltrametti E G and Bugajski S 1996 *J. Phys. A: Math. Gen.* **29** 247

[6] Orlov Y F 2002 *Phys. Rev.* A **65** 042106
[7] Christian J 2007 *Preprint* 0707.1333
[8] Basoalto R M and Percival I C 2003 *J. Phys. A: Math. Gen.* **36** 7411 and references therein
[9] Clauser J F, Horne M A, Shimony A and Holt R A 1969 *Phys. Rev. Lett.* **28** 880
[10] Bell J S 2004 *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press) chapter 4
[11] Accardi L and Regoli M 2000 *Preprint* quant-ph/0007005
[12] Matzkin A 2008 in preparation
[13] Santos E 1992 *Phys. Rev.* A **46** 3646
[14] Unruh W 2006 *Int. J. Quantum Inf.* **4** 209
[15] Goldstein H 1980 *Classical Mechanics* (Reading, MA: Addison-Wesley)